Section 9.6 Vector-Valued Functions

Reading Debrief

- Discuss Activity 9.6.2 (from Reading 3) w/ the person next to you. GeoGebra link
- Look at GeoGebra.
- Questions about anything from Section 9.6?

Activity 9.6.4 GeoGebra link

- Complete as a class.

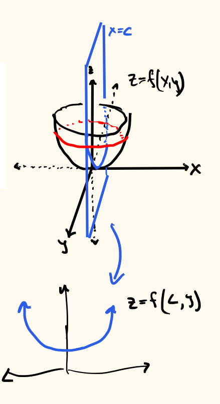
Activity 9.6.4. Consider the paraboloid defined by $f(x,y) = x^2 + y^2$.

a. Find a parameterization for the $x = 2$ trace of f. What type of curve does this trace describe?

b. Find a parameterization for the $y = -1$ trace of f. What type of curve does this trace describe?

c. Find a parameterization for the level curve $f(x,y) = 25$. What type of curve does this trace describe?

d. How do your responses change to all three of the preceding questions if you instead consider the function g defined by $g(x,y) = x^2 - y^2$? (Hint for generating one of the parameterizations: $\sec^2(t) - \tan^2(t) = 1$.)



(a) $z = f(2,y) = 4 + y^2$

$r(t) = \langle 2, t, t^2+4 \rangle$

★ If $f(x)$ is a function, its graph is a curve in $\mathbb{R}^2$. A parametrization of the graph is $r(t) = \langle t, f(t) \rangle$ ★

(b) $r(t) = \langle t, -1, 1+t^2 \rangle$

(c) $25 = f(x,y) = x^2 + y^2$ ⟷ circle w/ radius 5 at height $z=25$

$r(t) = \langle 5\cos(t), 5\sin(t), 25 \rangle$

A trace of $f(x,y)$ in the y-direction $(x=c)$ can be parametrized via

$$r(t) = \langle c, t, f(c,t) \rangle.$$

Similarly for traces in the x-direction $(y=c)$

$$r(t) = \langle t, c, f(t,c) \rangle.$$

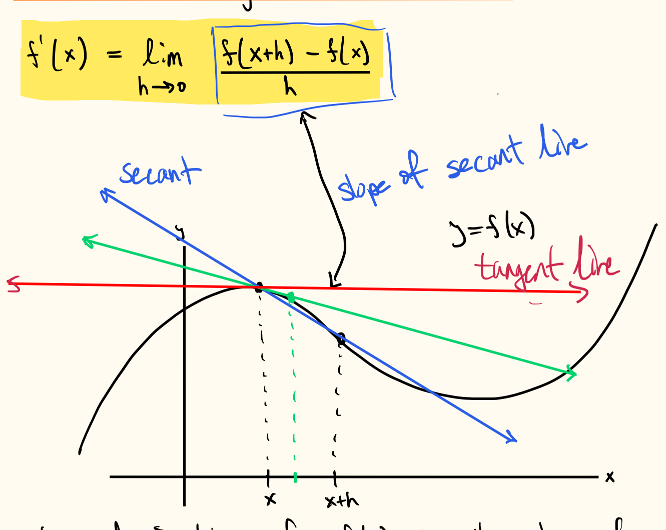
End of 9.6

Section 9.7 Derivatives of Vector Valued Functions

Derivative of single-variable functions

$$f'(x) = \lim_{h\to 0} \boxed{\frac{f(x+h)-f(x)}{h}}$$

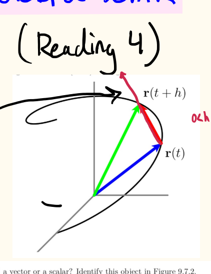
slope of secant line



The derivative of $f(x)$ is the slope of the tangent line.

Reading Debrief GeoGebra Link

- Discuss Activity 9.7.2 (Reading 4) w/ person next to you.
- Summary
- Look at GeoGebra.
- Questions?



a. Is the quantity $r(t+h) - r(t)$ a vector or a scalar? Identify this object in Figure 9.7.2.

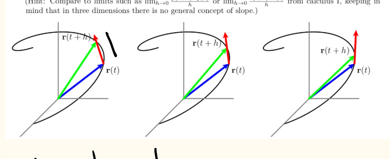
b. Is $\frac{r(t+h)-r(t)}{h}$ a vector or a scalar? Sketch a representative vector $\frac{r(t+h)-r(t)}{h}$ with $h<1$ in Figure 9.7.2.

c. Think of $r(t)$ as providing the position of an object moving along the curve these vectors trace out. What do you think that the vector $\frac{r(t+h)-r(t)}{h}$ measures? Why? (Hint: You might think analogously about difference quotients such as $\frac{f(x+h)-f(x)}{h}$ or $\frac{s(t+h)-s(t)}{h}$ from calculus I.)

d. Figure 9.7.3 presents three snapshots of the vectors $\frac{r(t+h)-r(t)}{h}$ as we let $h \to 0$. Write 2-3 sentences to describe key attributes of the vector

$$\lim_{h\to 0}\frac{r(t+h)-r(t)}{h}.$$

(Hint: Compare to limits such as $\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$ or $\lim_{h\to0}\frac{s(t+h)-s(t)}{h}$ from calculus I, keeping in mind that in three dimensions there is no general concept of slope.)



(a) vector, red arrow

(b) vector

(c) average velocity as the particle moves from $r(t)$ to $r(t+h)$

(d) • secant vectors approach tangent vector
- • length of red vector increases as $h \to 0$
- • points in the direction of motion of particle

Section 9.7.2 Computing Derivatives

Since limits of vector-valued functions are computed component-wise, so are derivatives?

Derivative of a Vector-Valued function

If $r(t) = \langle x(t), y(t), z(t) \rangle$, then

$$r'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

for all values of t at which x, y, z are differentiable. ▨

Activity 9.7.3

- Complete w/ your group.
- Class discussion.

Activity 9.7.3. For each of the following vector-valued functions, find $r'(t)$.

a. $r(t) = \langle \cos(t), t\sin(t), \ln(t) \rangle$.

b. $r(t) = \langle t^2 + 3t, e^{-2t}, \frac{t}{t^2+1} \rangle$.

c. $r(t) = \langle \tan(t), \cos(t^2), te^{-t} \rangle$.

d. $r(t) = \langle \sqrt{t^4+4}, \sin(3t), \cos(4t) \rangle$.

(a) $r'(t) = \langle \frac{d}{dt}\cos(t), \frac{d}{dt} t\sin t, \frac{d}{dt}\ln(t) \rangle$

$= \langle -\sin(t), \sin t + t\cos t, \frac{1}{t} \rangle$

chain rule: $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

product rule: $\frac{d}{dx}\left(f(x) \cdot g(x) \right) = f'(x)g(x) + g'(x)f(x)$

(b) $r'(t) = \left\langle 2t+3, -2e^{-2t}, \frac{1-t^2}{(t^2+1)^2} \right\rangle$

(c) $r'(t) = \langle \sec^2(t), -2t\sin(t^2), e^{-t} - te^{-t} \rangle$

(d) $r'(t) = \left\langle \frac{2t^3}{\sqrt{t^4+4}}, 3\cos(3t), -4\sin(4t) \right\rangle$

quotient rule: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'g - g'f}{g^2}$

Properties of the Derivative

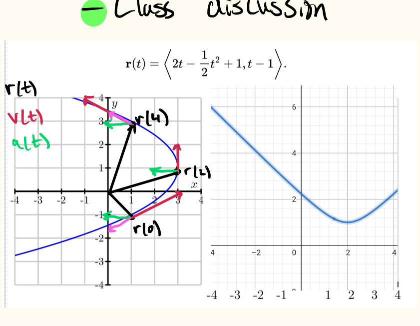
Properties of derivatives of vector-valued functions.

Let f be a differentiable real-valued function of a real variable t and let $\mathbf{r}$ and $\mathbf{s}$ be differentiable vector-valued functions of the real variable t. Then

1. $\frac{d}{dt}[\mathbf{r}(t) + \mathbf{s}(t)] = \mathbf{r}'(t) + \mathbf{s}'(t)$
2. $\frac{d}{dt}[f(t)\mathbf{r}(t)] = f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)$
3. $\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{s}(t)] = \mathbf{r}'(t) \cdot \mathbf{s}(t) + \mathbf{r}(t) \cdot \mathbf{s}'(t)$
4. $\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{s}(t)] = \mathbf{r}'(t) \times \mathbf{s}(t) + \mathbf{r}(t) \times \mathbf{s}'(t)$
5. $\frac{d}{dt}[\mathbf{r}(f(t))] = f'(t)\mathbf{r}'(f(t))$.

Activity 9.7.4

- Complete w/ your group.
- Class discussion



a. Find the object's velocity $\mathbf{v}(t)$.

b. Find the object's acceleration $\mathbf{a}(t)$.

c. Indicate on the left of Figure 9.7.4 the object's position, velocity and acceleration at the times $t = 0, 2, 4$. Draw the velocity and acceleration vectors with their tails placed at the object's position.

d. Recall that the speed is $|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$. Find the object's speed and graph it as a function of time t on the right of Figure 9.7.4. When is the object's speed the slowest? When is it increasing? When is it decreasing?

e. What seems to be true about the angle between $\mathbf{v}$ and $\mathbf{a}$ when the speed is at a minimum? What is the angle between $\mathbf{v}$ and $\mathbf{a}$ when the speed is increasing? when the speed is decreasing?

f. Since the square root is an increasing function, we see that the speed increases precisely when $\mathbf{v} \cdot \mathbf{v}$ is increasing. Use the product rule for the dot product to express $\frac{d}{dt}(\mathbf{v} \cdot \mathbf{v})$ in terms of the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$. Use this to explain why the speed is increasing when $\mathbf{v} \cdot \mathbf{a} > 0$ and decreasing when $\mathbf{v} \cdot \mathbf{a} < 0$. Compare this to part (d).

g. Show that the speed's rate of change is

$$\frac{d}{dt}|\mathbf{v}(t)| = \text{comp}_{\mathbf{v}}\mathbf{a}.$$

(a) $v(t) = r'(t) = \langle 2-t, 1 \rangle$

(b) $a(t) = v'(t) = r''(t) = \langle -1, 0 \rangle$

(c) $r(0) = \langle 1, -1 \rangle$ $r(2) = \langle 3, 1 \rangle$ $r(4) = \langle 1, 3 \rangle$

• $v(0) = \langle 2, 1 \rangle$ $v(2) = \langle 0, 1 \rangle$ $r(4) = \langle -2, 1 \rangle$

(d) $|v| = \sqrt{v \cdot v} = \sqrt{t^2 - 4t + 5}$

Slowest: $t = 2$

Increasing: $4 > t > 2$

Decreasing: $-4 < t < 2$

(e) $v(2)$ and $a(2)$ are $\perp$

For $2 < t < 4$, $v(t)$ and $a(t)$ are acute

For $-4 < t < 2$, $v(t)$ and $a(t)$ are obtuse.

(f) $\frac{d}{dt}(v \cdot v) = v' \cdot v + v \cdot v'$

$= 2v \cdot v'$

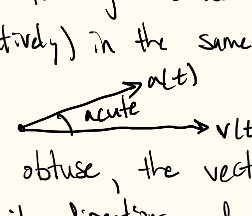
$= 2(v \cdot a)$

speed $= |v|$ is increasing $\iff$ $v \cdot v$ increasing

$\iff \frac{d}{dt}(v \cdot v) > 0$

$\iff$ angle between $v(t)$ and $a(t)$ is acute.

In other words, when the angle is acute, v and a are pointing (relatively) in the same direction.



When the angle is obtuse, the vectors point (relatively) in opposite directions and $|v|$ is decreasing.

(g) $\frac{d}{dt}(|v(t)|) = \frac{d}{dt}\sqrt{v \cdot v}$

$= \frac{1}{2} \cdot \frac{1}{\sqrt{v \cdot v}} \cdot (2v \cdot a)$

$= \frac{v \cdot a}{|v|} = \text{comp}_v a$

$(|\text{proj}_v a| = |\text{comp}_v(a)|)$

Section 9.7.3 Tangent Lines

Every line is determined by a point on the line and a direction vector.

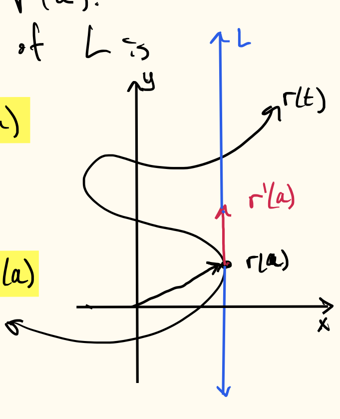
The tangent line L to a curve $r(t)$ at $t = a$ is the line w/ initial pt. $r(a)$ and direction $r'(a)$.

So a parametrization of L is

$$L_1(t) = r(a) + t\,r'(a)$$

or

$$L_2(t) = r(a) + (t-a)\,r'(a)$$



Activity 9.7.5 GeoGebra link

- Complete and discuss w/ your group
- Class discussion

Activity 9.7.5. Let

$$\mathbf{r}(t) = \cos(t)\mathbf{i} - \sin(t)\mathbf{j} + t\mathbf{k}.$$

Sketch the curve using some appropriate tool.

a. Determine the coordinates of the point on the curve traced out by $\mathbf{r}(t)$ when $t = \pi$.

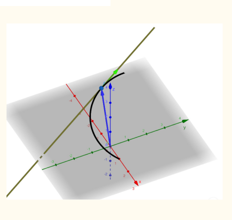
b. Find a direction vector for the line tangent to the graph of $\mathbf{r}$ at the point where $t = \pi$.

c. Find the parametric equations of the line tangent to the graph of $\mathbf{r}$ when $t = \pi$.

d. Sketch a plot of the curve $\mathbf{r}(t)$ and its tangent line near the point where $t = \pi$. In addition, include a sketch of $\mathbf{r}'(\pi)$. What is the important role of $\mathbf{r}'(\pi)$ in this activity?

(a) $r(\pi) = \langle -1, 0, \pi \rangle$ blue arrow

(b) $r'(\pi) = \langle 0, 1, 1 \rangle$ green arrow



(c) $L(t) = r(\pi) + t\,r'(\pi)$

$= \langle -1, 0, \pi \rangle + t\langle 0, 1, 1 \rangle$

$= \langle -1, 0, \pi \rangle + \langle 0, t, t \rangle$

$= \langle -1, t, t+\pi \rangle$

Section 9.7.4 Integrating a Vector-Valued Function

Recall: an antiderivative of a single-variable function $f(x)$ is a function $F(x)$ s.t.

$$\frac{d}{dx}F(x) = f(x).$$

The indefinite integral $\int f(x)\,dx$ is the general antiderivative of f. From Fundamental Thm of Calculus, we have

$$\int f(x)\,dx = F(x) + C.$$

Definition 9.7.5 An antiderivative of a vector-valued function $r(t)$ is a vector-valued function $R(t)$ such that $R'(t) = r(t)$.

The indefinite integral $\int r(t)\,dt$ is the general antiderivate of $r(t)$. ▨

Since we differentiate componentwise, we also integrate componentwise

$$\int r(t)\,dt = \left\langle \int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt \right\rangle$$

If $R(t)$ is an antiderivative of $r(t)$

$$\int r(t)\,dt = R(t) + v$$

where v is a "constant" vector.

Activity 9.7.6

- Complete w/ your group.
- Class discussion.

Activity 9.7.6. Suppose a moving object in space has its velocity given by

$$\mathbf{v}(t) = (-2\sin(2t))\mathbf{i} + (2\cos(t))\mathbf{j} + \left(1 - \frac{1}{1+t}\right)\mathbf{k}.$$

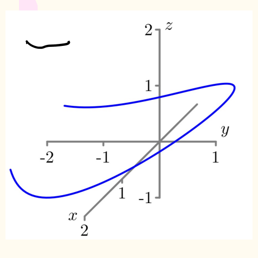
A graph of the position of the object for times t in $[-0.5, 3]$ is shown in Figure 9.7.6. Suppose further that the object is at the point $(1.5, -1, 0)$ at time $t = 0$.

a. Determine $\mathbf{a}(t)$, the acceleration of the object at time t.

b. Determine $\mathbf{r}(t)$, position of the object at time t.

c. Compute and sketch the position, velocity, and acceleration vectors of the object at time $t = 1$, using Figure 9.7.6.

d. Finally, determine the vector equation for the tangent line, $\mathbf{L}(t)$, that is tangent to the position curve at $t = 1$.



Skip (a), (c), (d)

(b) $\int v(t)\,dt = \left\langle \int -2\sin(2t)\,dt, \int 2\cos(t)\,dt, \int 1 - \frac{1}{1+t}\,dt \right\rangle$

$= \langle \cos(2t) + c_1, 2\sin(t) + c_2, t - \ln(|1+t|) + c_3 \rangle$

$= \langle \cos(2t), 2\sin(t), t - \ln(|1+t|) \rangle + \langle c_1, c_2, c_3 \rangle$

End of Section 9.7